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# Thick Beam Theory and Finite Element Model with Zig-Zag Sublaminate Approximations

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**A new technical theory and associated finite element model is introduced for thick laminated and sandwich beams. The theory can be cast as a layerwise theory with high-order zig-zag sublaminate approximations, thus greatly reducing the number of degrees of freedom required to accurately describe the bending and transverse shear kinematics in thick laminates. Furthermore, the theory is adaptable, allowing the user to choose the number of sublaminate approximations to achieve the desired accuracy. Based on this theory, a simple, efficient, and robust finite element model is developed that has the nodal topology of a four-noded planar element yet has the advantages of beam-type kinematics and a special interpolation scheme that obviates locking. The element contains a single zig-zag sublaminate approximation. If desired, multiple elements can be used through the thickness of a laminate to increase accuracy.**

## I. Introduction

MANY of the current military and commercial applications of composite materials call for thick-section laminates composed of over 100 layers or sandwich panels with a honeycomb core between face sheets of laminated composite. A primary concern in the analysis of thick-section laminates and sandwich panels is the recognition that transverse stresses and deformations may not be neglected as they often are in thin laminates. The commonly assumed linear variation of in-plane displacement and stress components through the thickness of the laminate, which is valid in an average sense, is not adequate for accurately predicting the response of these structures.

Because three-dimensional analysis of thick laminated structures is often computationally intractable, it is highly desirable to have a shell-type model that captures the important local through-thickness variations of displacements and stresses in thick laminates while maintaining the efficiency and convenience of a shell-type model. For the types of problems just listed, inclusion of first-order transverse shear effects[1] is often not adequate, and higher order effects need to be taken into account.

Two approaches have been taken in the development of laminate theories that incorporate higher order effects in thick laminates—the equivalent single-layer approach and the discrete-layer approach. In equivalent single-layer theories, high-order terms are maintained in the expansion of the displacement components in the thickness coordinate (see, for example, Refs. 2–5). In this way, nonlinear variations of displacements, strains, and stresses through the thickness of the laminate are permitted along with an approximation of transverse normal deformations and/or transverse shear deformations. These theories all provide improvements over the first-order shear deformation theory but also have a common drawback. The thickness variation of displacements, and thus strains, is assumed to be continuous and smooth (i.e., $C^1$ continuous). This characteristic precludes the satisfaction of transverse stress continuity at interfaces between adjacent layers of different materials and does not accurately reflect the kinematics in laminates that contain adjacent plies with drastically different constitutive properties.

To overcome the drawbacks of equivalent single-layer theories, discrete-layer (or layerwise) theories have been developed in which a unique displacement field is assumed within each layer. Theories of this type may be classified into two groups according to whether or not the number of degrees of freedom (DOFs) in the theory is dependent on the number of layers in the laminate. When the number of DOFs is coupled to the number of layers (see, for example, Refs. 6–8), the computational effort required in the analysis of thick-section laminates is comparable to that required by a fully three-dimensional analysis. A few discrete-layer theories have been developed that contain a constant number of DOFs irrespective of the number of layers in the laminate (see, for example, Refs. 9–18). In these theories, the additional DOFs are eliminated by enforcing continuity of the transverse shear stress components at the interfaces between each layer and by satisfying the zero shear traction conditions on the top and bottom surfaces of the laminate.

Comparison of results obtained with fully three-dimensional continuum-based theories demonstrates that discrete-layer theories with a constant number (independent of the number of layers) of DOFs, often called zig-zag theories, provide very accurate approximations of structural response as well as through-thickness variations of in-plane displacements and stresses for thick laminates provided the number of layers is not too large. However, it has been found that as the number of layers in the laminate and/or the complexity of the lamination scheme is increased, the error in the analysis may increase to unacceptable levels. In these cases, it becomes necessary to resort to the more general layerwise theories with independent high-order displacement approximations for each layer.

In the present paper, a new laminated beam theory is developed that can be considered a hybrid theory, combining the general layerwise theories with independent layerwise kinematic approximations and the efficient zig-zag theories in which the layerwise DOFs are eliminated by enforcing transverse shear stress continuity conditions. In the new theory, the through-the-thickness approximation of the in-plane displacement components takes the form of a layerwise theory in which each layer is really a sublaminate containing several, even many, physical layers. Within each sublaminate, a zig-zag through-the-thickness approximation of the in-plane displacement components is taken in which the layerwise DOFs are eliminated by enforcing continuity of transverse stresses. Shear traction conditions at the top and bottom of each sublaminate are also satisfied. The theory includes the effects of transverse normal strain and is cast in a form that is exceptionally well suited for solution by the finite element method. An accurate and convenient four-noded planar element with beam-type kinematics is developed and its utility is demonstrated.

## II. Formulation of the Laminated Beam Theory

In the present theory, the layerwise construction of laminated composite beams is modeled as $M$ sublaminates, with each

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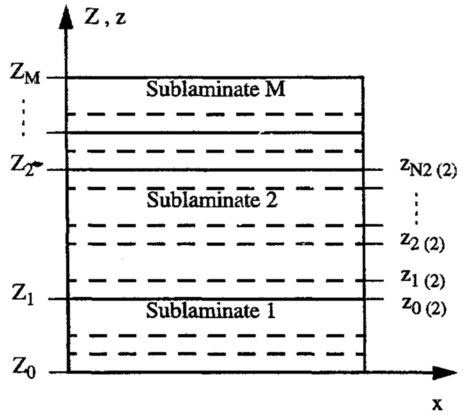


**Fig. 1 Schematic of sublaminate and layer divisions in the laminate theory.**

sublaminate containing $N_m$ layers, where $m$ is the sublaminate number (see Fig. 1). The total number of layers in the laminate is then

$$N_{\text{total}} = \sum_{m=1}^{M} N_m \tag{1}$$

The thickness and material stiffness properties of each layer are arbitrary, and it is assumed that adjacent layers are perfectly bonded together. In the following, $z$ is a local (sublaminate) thickness coordinate with its origin at the bottom of the sublaminate, whereas $Z$ is the global (laminate) thickness coordinate. The coordinate $x$ is measured along the length of the beam.

If the width of the beam is small, then $\sigma_y$, $\tau_{xy}$, and $\tau_{zy}$ are negligible, and a state of plane stress can be assumed. The plane stress constitutive relations for the $k$th layer of the beam take the form

$$\begin{bmatrix} \sigma_x \\ \sigma_z \\ \tau_{xz} \end{bmatrix}^{(k)} = \begin{bmatrix} C_{11}^{(k)} & C_{13}^{(k)} & 0 \\ C_{13}^{(k)} & C_{33}^{(k)} & 0 \\ 0 & 0 & C_{55}^{(k)} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_z \\ \gamma_{xz} \end{bmatrix}^{(k)} \tag{2}$$

The infinitesimal strain-displacement relations are

$$\varepsilon_x^{(k)} = \frac{\partial u_x^{(k)}}{\partial x} \qquad \varepsilon_z^{(k)} = \frac{\partial u_z^{(k)}}{\partial z}$$
$$\gamma_{xz}^{(k)} = \frac{\partial u_z^{(k)}}{\partial x} + \frac{\partial u_x^{(k)}}{\partial z} \tag{3}$$

In each sublaminate, an independent displacement field is assumed in which the through-the-thickness variation of in-plane displacements is described by a cubic polynomial in the local thickness coordinate with a piecewise linear (or zig-zag) function superimposed upon it. The transverse deflection is assumed to vary linearly with the transverse coordinate. The displacement components of the $n$th layer within the $m$th sublaminate, where $1 \le n \le N_m$ and $1 \le m \le M$, can be written as

$$u_x^{(m,n)} = \sum_{k=0}^{3} z^k \hat{u}_k^{(m)} + \sum_{i=1}^{n-1} (z - z_i)\xi_i^{(m)}$$
$$u_z^{(m,n)} = w_b^{(m)}\left[1 - (z/h_m)\right] + w_t^{(m)}(z/h_m) \tag{4}$$

where the subscripts $b$ and $t$ refer to the bottom and top surfaces, respectively, of the $m$th sublaminate, and $h_m$ is the total thickness of the $m$th sublaminate.

It is possible to eliminate the DOFs $\xi_i^{(m)}$ in Eqs. (4) by enforcing the condition of transverse shear stress continuity at each layer interface. The condition of shear stress continuity at the $j$th interface is

$$\tau_{xz}^{(m,j)}\Big|_{z=z_j} = \tau_{xz}^{(m,j+1)}\Big|_{z=z_j} \tag{5}$$

Making use of Eqs. (2–4) in Eq. (5), it is found that

$$\xi_i = \hat{a}_i\left(\hat{u}_1 + \frac{\mathrm{d}w_b}{\mathrm{d}x}\right) + \hat{b}_i\hat{u}_2 + \hat{c}_i\left(\frac{\mathrm{d}w_t}{\mathrm{d}x} - \frac{\mathrm{d}w_b}{\mathrm{d}x}\right) + \hat{d}_i\hat{u}_3 \tag{6}$$

where

$$\hat{a}_i = \left[\frac{C_{55}^{(i)}}{C_{55}^{(i+1)}} - 1\right]\left(1 + \sum_{j=1}^{i-1} \hat{a}_j\right)$$
$$\hat{b}_i = \left[\frac{C_{55}^{(i)}}{C_{55}^{(i+1)}} - 1\right]\left(2z_i + \sum_{j=1}^{i-1} \hat{b}_j\right)$$
$$\hat{c}_i = \left[\frac{C_{55}^{(i)}}{C_{55}^{(i+1)}} - 1\right]\left(\frac{z_i}{h_m} + \sum_{j=1}^{i-1} \hat{c}_j\right) \tag{7}$$
$$\hat{d}_i = \left[\frac{C_{55}^{(i)}}{C_{55}^{(i+1)}} - 1\right]\left(3z_i^2 + \sum_{j=1}^{i-1} \hat{d}_j\right)$$

Additional simplification of Eqs. (4) can be achieved through satisfaction of the transverse shear traction boundary conditions at the top and bottom surfaces of each sublaminate. For sublaminate $m$, these conditions are

$$\tau_{xz}^{(m,1)}\Big|_{z=0} = \tau_b^{(m)}, \qquad \tau_{xz}^{(m,N_m)}\Big|_{z=h_m} = \tau_t^{(m)} \tag{8}$$

where $\tau_b^{(m)}$ and $\tau_t^{(m)}$ are the applied shear tractions (or interlaminar shear stresses, as the case may be) at the bottom and top surfaces, respectively, of the $m$th sublaminate.

The displacement field is cast in its final form by introducing the variables

$$u_b^{(m)} = u_x^{(m,1)}\Big|_{z=0} = \hat{u}_0^{(m)}, \qquad u_t^{(m)} = u_x^{(m,N_m)}\Big|_{z=h_m} \tag{9}$$

Equations (8) and (9) can be solved either analytically or numerically so that the displacement field for the $m$th sublaminate can be expressed in terms of the operative DOFs (all functions of $x$ only):

$$u_b^{(m)}, w_b^{(m)}, \theta_b^{(m)}, \tau_b^{(m)}, u_t^{(m)}, w_t^{(m)}, \theta_t^{(m)}, \tau_t^{(m)} \tag{10}$$

where again the subscripts $b$ and $t$ refer to the bottom and top surfaces, respectively, of the sublaminate, and

$$\theta_b^{(m)} = \frac{\mathrm{d}w_b^{(m)}}{\mathrm{d}x}, \qquad \theta_t^{(m)} = \frac{\mathrm{d}w_t^{(m)}}{\mathrm{d}x} \tag{11}$$

The displacement field now takes the form

$$u_x^{(m,n)} = u_b^{(m)}\left[1 + p_1^{(m)}z^2 + p_2^{(m)}z^3 + \sum_{i=1}^{n-1}(z - z_i)a_i^{(m)}\right]$$
$$+ u_t^{(m)}\left[-p_1^{(m)}z^2 - p_2^{(m)}z^3 - \sum_{i=1}^{n-1}(z - z_i)a_i^{(m)}\right]$$
$$+ \theta_b^{(m)}\left[-z + p_3^{(m)}z^2 + p_4^{(m)}z^3 + \sum_{i=1}^{n-1}(z - z_i)b_i^{(m)}\right]$$
$$+ \theta_t^{(m)}\left[p_5^{(m)}z^2 + p_6^{(m)}z^3 + \sum_{i=1}^{n-1}(z - z_i)c_i^{(m)}\right]$$
$$+ \tau_b^{(m)}\left[p_7^{(m)}z + p_8^{(m)}z^2 + p_9^{(m)}z^3 + \sum_{i=1}^{n-1}(z - z_i)d_i^{(m)}\right]$$
$$+ \tau_t^{(m)}\left[p_{10}^{(m)}z^2 + p_{11}^{(m)}z^3 + \sum_{i=1}^{n-1}(z - z_i)e_i^{(m)}\right] \tag{12}$$
$$u_z^{(m,n)} = w_b^{(m)}[1 - (z/h_m)] + w_t^{(m)}(z/h_m)$$

In Eq. (12), $p_k$ ($k = 1$–$11$) and $a_i$, $b_i$, $c_i$, $d_i$, and $e_i$ are functions of the layer shear stiffnesses and thicknesses and can thus be calculated a priori. Their functional forms are given in the Appendix.

In the sublaminate displacement field just described, the functions of $z$ that multiply the DOFs (10) can be viewed as shape functions that describe the through-the-thickness variation of these measures within the sublaminate. Thus, a layerwise laminate theory can be developed in the form introduced by Reddy,[6] where now each layer in the layerwise theory may contain several, even many, physical layers. Of course, this concept was always possible to envision with the layerwise theories, but the present formulation provides an extremely accurate and efficient approximation within each sublaminate. The sublaminate approximations are connected by imposing continuity of displacements and transverse shear stresses at the interface between each sublaminate region. The theory thus has the following properties: 1) continuity of transverse shear stresses through the entire thickness of the laminate is satisfied; 2) shear traction boundary conditions on the top and bottom surfaces of the laminate are satisfied exactly; 3) a piecewise (layerwise) continuous through-the-thickness variation of the in-plane displacements is allowed, yet the number of DOFs in each sublaminate is independent of the number of layers in that sublaminate; 4) a linear through-the-thickness variation of the transverse deflection in each sublaminate explicitly accounts for transverse normal strains and stresses; and 5) only engineering-type DOFs are used—displacements and rotations, plus shear traction terms that are always known on the surfaces and unknown between sublaminates.

The accuracy and efficiency of the present theory is thus adaptable, depending on the number of sublaminates chosen to model a given laminate. If only one sublaminate is used through the thickness of the entire laminate, then the theory falls into the class of zig-zag theories such as those in Refs. 9–18. If the number of sublaminates is equal to the number of layers in the laminate, then the theory can be categorized with the more general layerwise theory of Reddy.[6] In this case, a cubic layerwise through-the-thickness variation of the in-plane displacement components would yield extremely high accuracy (as in Ref. 8). The optimal number of sublaminate approximations required for accurate analysis of thick multilayered composite laminates or sandwich panels will often be greater than one but far less than the number of layers in the laminate. The adaptable nature of the current theory can thus be used to great computational advantage.

## III. Finite Element Model

Because first derivatives of $w_b$ and $w_t$ appear in the displacement field (12), second derivatives of these variables are present in the strain energy functional, requiring them to be $C^{(1)}$ continuous. In the current finite element model, this continuity requirement is weakened by treating $\theta_b$ and $\theta_t$ as independent DOFs and subsequently imposing the constraints

$$g_b = \theta_b - \frac{\mathrm{d}w_b}{\mathrm{d}x} = 0, \qquad g_t = \theta_t - \frac{\mathrm{d}w_t}{\mathrm{d}x} = 0 \tag{13}$$

via a combination of an interdependent interpolation scheme and the penalty method.[19–21] All DOFs in the displacement field are then required to be only $C^0$ continuous.

The principle of minimum total potential energy is employed to develop the finite element model. For a constrained system, we have

$$\delta\Pi_p = \delta U + \delta V + \delta\left[\frac{\gamma_b}{2}\left(\int_\Omega g_b^2\right)\mathrm{d}x\right] + \delta\left[\frac{\gamma_t}{2}\left(\int_\Omega g_t^2\right)\mathrm{d}x\right] = 0 \tag{14}$$

where $U$ is the internal strain energy, $V$ is the potential energy of external forces, and $\gamma_b$ and $\gamma_t$ are penalty parameters that enforce constraints (13).

Substituting Eqs. (2), (3), (12), and (13) into Eq. (14), the governing equations, boundary conditions, and displacement-based finite element model can be developed. Although the finite element geometry could take the form of a two-noded beam element with eight DOFs per node, it is advantageous to use the topology of a four-noded, planar element, as shown in Fig. 2 (bottom). This topology allows the laminate thickness to be conveniently subdivided and modeled by multiple finite elements (representing sublaminates). It is thus possible to increase the accuracy of the finite element model, as needed, to capture through-the-thickness gradients and transverse (interlaminar) stresses. It is also possible to simulate delaminations using the redundant node concept.

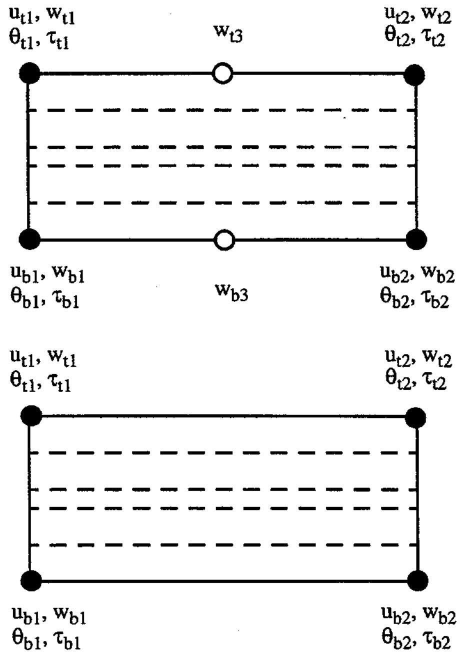


**Fig. 2 Nodal topology of unconstrained element (top) and constrained element (bottom).**

The element formulation takes advantage of an interdependent interpolation concept introduced by Tessler and Dong[22] and recently used by other element developers.[17,18,23] Except for the element interpolation scheme discussed next, all aspects of the finite element formulation follow the well-known standard procedures (see, for example, Ref. 24), and the details are omitted.

In the constrained element formulation, the transverse deflection DOFs $w_b$ and $w_t$ are initially approximated using quadratic Lagrange interpolation functions, whereas all other DOFs are expanded using linear Lagrange interpolation functions. Such an approximation scheme results in a six-noded element in which each of the two midside nodes contains a single DOF associated with the transverse deflection (see Fig. 2 top). These midside nodes are eliminated by considering a modified form of the constraint conditions in Eq. (13):

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\theta_b - \frac{\mathrm{d}w_b}{\mathrm{d}x}\right) = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}x}\left(\theta_t - \frac{\mathrm{d}w_t}{\mathrm{d}x}\right) = 0 \tag{15}$$

By substituting the element approximations into Eq. (15), the midside DOFs $w_{b3}$ and $w_{t3}$ may be determined in terms of the other nodal DOFs, resulting in a new interdependent element interpolation scheme:

$$\begin{aligned} w_b &\cong \sum_{j=1}^{2} w_{bj}P_j + \sum_{j=1}^{2}\theta_{bj}\hat{N}_j \\ w_t &\cong \sum_{j=1}^{2} w_{tj}P_j + \sum_{j=1}^{2}\theta_{tj}\hat{N}_j \end{aligned} \tag{16}$$

where $P_1$ and $P_2$ are the linear Lagrange interpolation functions,

$$\hat{N}_1 = (a_e/8)(1-\xi^2) \qquad \hat{N}_2 = -(a_e/8)(1-\xi^2)$$

where $a_e$ is the length of element $e$, and $\xi$ is the natural axial coordinate in the element ($-1 \le \xi \le 1$). If the constraints in Eqs. (13)

are viewed as being composed of a constant part and a linear (in $x$) part, then the preceding interdependent interpolation scheme satisfies the linear part of the constraints (13) identically. This makes the penalty function technique in the present formulation more robust, because the penalty parameter enforces only the constant part of the constraints. This approach effectively increases the order of the element without introducing any additional nodes or DOFs and eliminates the shear locking problem so that an exact order of integration may be used. A consistent force vector is also obtained.

## IV. Numerical Results

Numerical results are presented for bending of a simply supported laminated beam subjected to a sinusoidally varying transverse load of magnitude one (see Fig. 3). Comparisons are made between predictions of models based on first-order shear deformation theory (FSDT),[1] the present layerwise zig-zag theory (LZZT), and an exact elasticity solution.[25]

The material properties used in the analyses are as listed in Table 1, where $E_0$ is a reference value that is chosen here to be $1.0 \times 10^6$. Laminates composed of combinations of these materials have been used previously[7] and provide a good test of the models because they are distinctly different in their stiffness characteristics. Material 1 is considered compliant in tension/compression and compliant in shear. Material 2 is stiff in tension/compression and stiff in shear. Material 3 is stiff in tension/compression and compliant in shear.

The laminate configuration used in the present numerical study is described in Table 2. It is an unsymmetrical five-layer laminate with seemingly random layer thicknesses and material properties that vary drastically from layer to layer. This example provides a very rigorous test of a laminate theory's ability to model complex laminates subjected to sharply varying loads. Although results are presented here for only one laminate, it should be noted that the current model has been tested on many problems with a wide variety of lamination sequences and geometries, including sandwich beams. In each case, comparisons between predictions of the present model and the exact elasticity solution were at least as good as the comparisons presented here.

For a thick laminate with span-to-thickness ratio of 4, the normalized maximum deflection and the normalized maximum in-plane normal stress predicted using a single sublaminate are plotted in Fig. 4 vs the number of elements used in the mesh. The results are normalized by the exact solution of Pagano,[25] which has been modified for beam analysis by specifying the appropriate constitutive equations for beams (2) instead of those for plates. When only four elements are used, the error in both cases is less than 4%. However, it appears that a nearly converged solution is attained using approximately 10 elements. Thus, all subsequent finite element results are obtained using a uniform mesh of 10 four-noded elements (with full integration) for LZZT and 10 two-noded elements (with interdependent interpolation[22]) for FSDT. Note in Fig. 4 that there is approximately 5% error in the predicted stress when 10 elements are used. This error exists because the current model does not predict the exact result for this laminate when $L/h = 4$. Thus, the error is primarily due to assumptions in the theory as opposed to finite element approximation errors. The accuracy of these predictions increases rapidly as the aspect ratio of the beam increases.

**Table 1 Material properties used in the numerical analyses**

| Material | $E/E_0$ | $G/E_0$ |
|---|---|---|
| 1 | 1.0 | 0.2 |
| 2 | 32.5 | 8.2 |
| 3 | 25.0 | 0.5 |

**Table 2 Description of laminate sequence**

| Laminate | Layer no. | Relative thickness, volume fraction | Material no. |
|---|---|---|---|
| Five-layer random | 1 | 0.100 | 1 |
| | 2 | 0.250 | 2 |
| | 3 | 0.150 | 3 |
| | 4 | 0.200 | 1 |
| | 5 | 0.300 | 3 |

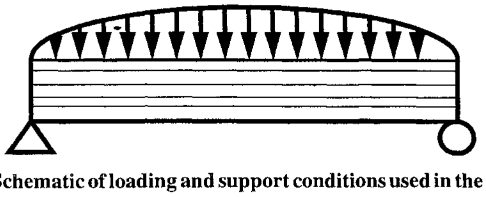

**Fig. 3 Schematic of loading and support conditions used in the example problems.**

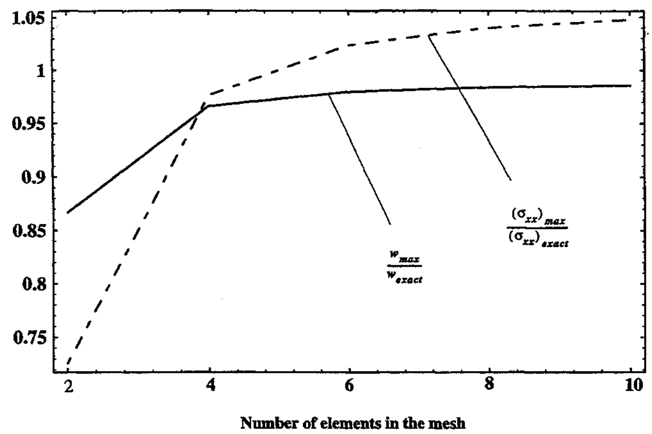


**Fig. 4 Predicted normalized maximum deflection and normalized maximum inplane normal stress vs number of elements in the mesh.**

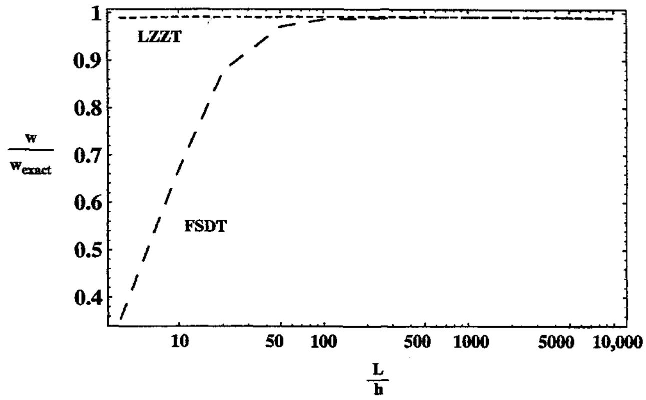


**Fig. 5 Normalized center deflection vs span-to-thickness ratio of a five-layer simply supported beam subjected to a sinusoidal load.**

In Fig. 5, the predicted normalized center deflection vs the span-to-thickness ratio of the beam is shown. The deflection predictions of FSDT degrade sharply for aspect ratios less than about 50, largely because the material properties of adjacent layers vary drastically. It should be pointed out that a shear correction factor of $\frac{5}{6}$ (valid only for isotropic beams) was used to obtain the FSDT results, which accounts for some of the error. Predictions obtained using LZZT with one element (one sublaminate) through the thickness are accurate for all aspect ratios greater than about two, and no shear correction factor is needed. Because of the interpolation schemes used, there is no locking in either model as the beam thickness (and thus the thickness of each element) decreases.

In Figs. 6 and 7, the through-thickness distributions of in-plane displacement and in-plane normal stress, respectively, as predicted by FSDT, LZZT, and elasticity, are plotted for a thick laminate having a span-to-thickness ratio of 4. It can be seen that LZZT, with only one element through the thickness, does an excellent job of predicting the through-thickness distributions of both in-plane displacements and stresses, even for this very thick laminate. FSDT is not able to capture these variations. These two plots highlight the

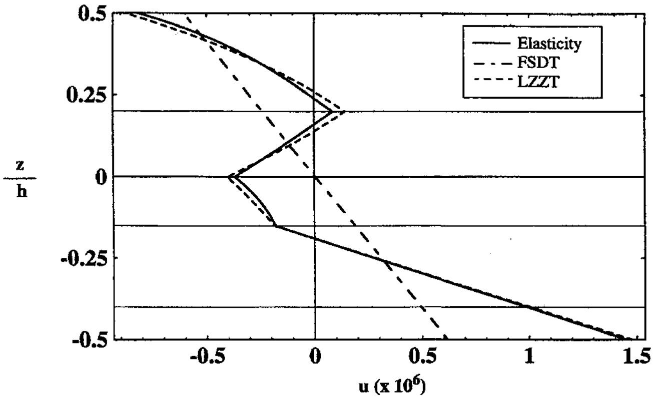


Fig. 6 Axial displacement vs normalized thickness coordinate at the end of a simply supported beam subjected to a sinusoidal load.

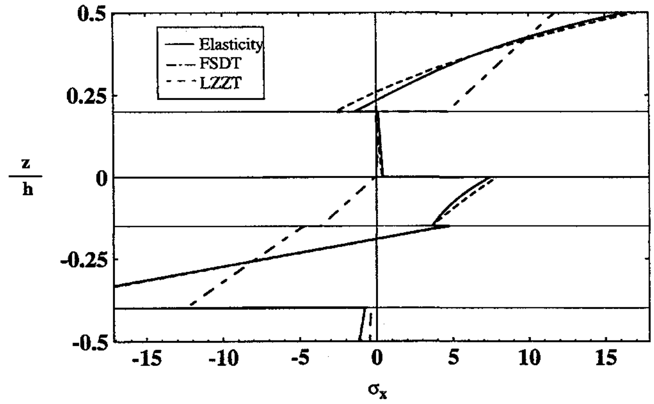


Fig. 7 Axial stress vs normalized thickness coordinate at the midspan of a simply supported beam subjected to a sinusoidal load.

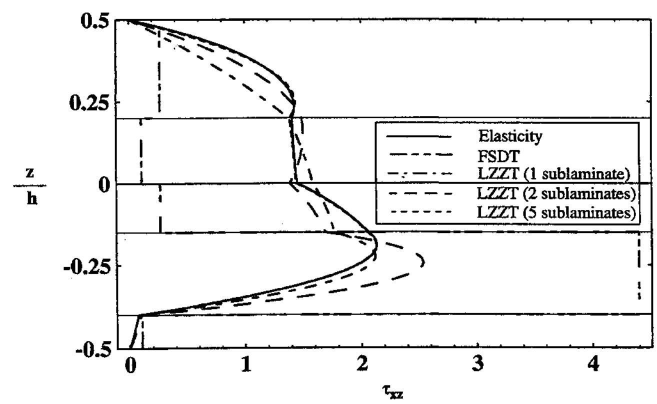


Fig. 8 Transverse shear stress vs normalized thickness coordinate at $x = L/10$ of a simply supported beam subjected to a sinusoidal load.

importance of including the zig-zag through-thickness variation of in-plane displacements (and, hence, in-plane strains) for laminates in which the material properties of adjacent layers vary considerably.

In Figs. 8 and 9, the utility of being able to use multiple LZZT elements through the thickness of a laminate is demonstrated for the prediction of transverse shearing stresses and the through-thickness distribution of the transverse deflection in a thick laminate with span-to-thickness ratio of 4. When five LZZT elements (one per physical layer) are used through the thickness, the predictions of transverse shearing stress [calculated using the constitutive equations (2)] are indistinguishable from the exact solution, and the variation of transverse deflection is very good. Because increasing the number of sublaminates in the model also increases the finite element discretization through the thickness of the laminate, a greater number of DOFs is required for such refined analyses. It has been shown, however, that excellent results can usually be obtained using only

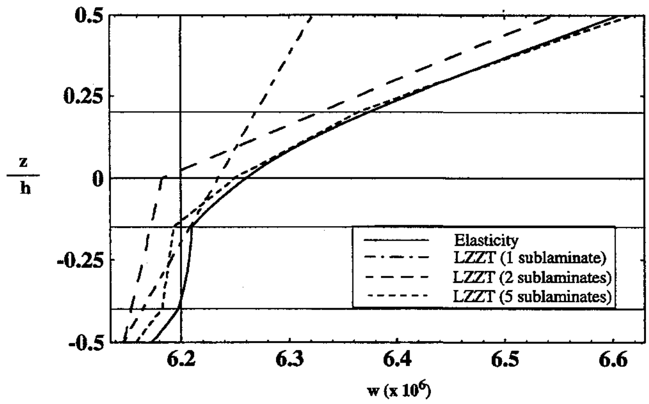


Fig. 9 Transverse deflection vs normalized thickness coordinate at midspan of a simply supported beam subjected to a sinusoidal load.

one element through the thickness of a laminate. Further, because in-plane normal stresses are predicted very accurately, transverse stresses can be obtained by the now standard approach of integrating the two-dimensional equations of equilibrium through the thickness (as in Ref. 18).

## V. Conclusions

The new theory and finite element model described herein show excellent promise for accurate, efficient, and convenient modeling of laminated and sandwich beams. The level of accuracy in the prediction of through-the-thickness variations of displacements, strains, and stresses can be varied by choosing the number of sublaminates used in the model. In most cases, only one sublaminate, or one finite element through the thickness, is needed to achieve the desired accuracy of both global deflections and local in-plane stresses, even for very thick laminates.

## Appendix: Definition of Functions in Eq. (12)

$$p_1 = 1/\bar{c}, \qquad p_2 = \bar{b}/\bar{c}, \qquad p_3 = -(\bar{a}/\bar{c})$$

$$p_4 = \tilde{c} + \bar{b}(\tilde{a}/\bar{c}), \qquad p_5 = -(\tilde{b}/\bar{c}), \qquad p_6 = -p_4 \tag{A1}$$

$$p_7 = 1/C_{55}^{(1)}, \qquad p_8 = -(\bar{d}/\bar{c}), \qquad p_9 = -\left(\bar{a}/C_{55}^{(1)}\right) + \bar{b}(\bar{d}/\bar{c})$$

$$p_{10} = -(\bar{e}/\bar{c}), \qquad p_{11} = \left(\bar{d}/C_{55}^{(N_m)}\right) + \bar{b}(\bar{e}/\bar{c})$$

$$a_i = \frac{\hat{d}_i\bar{b} - \hat{b}_i}{\bar{c}}, \qquad b_i = \tilde{a}a_i + \hat{d}_i\tilde{c} - \hat{c}_i$$

$$c_i = \tilde{b}a_i - \hat{d}_i\tilde{c} + \hat{c}_i, \qquad d_i = \bar{d}a_i - \frac{\hat{d}_i\bar{a} - \hat{a}_i}{C_{55}^{(1)}} \tag{A2}$$

$$e_i = \bar{e}a_i + \frac{\hat{d}_i\bar{d}}{C_{55}^{(N_m)}}$$

$$\bar{a} = \bar{d}\left(1 + \sum_{i=1}^{N_m-1}\hat{a}_i\right), \qquad \bar{b} = \bar{d}\left(2h_m + \sum_{i=1}^{N_m-1}\hat{b}_i\right)$$

$$\bar{c} = \bar{d}\left(1 + \sum_{i=1}^{N_m-1}\hat{c}_i\right) \qquad \bar{d} = \frac{1}{3h_m^2 + \sum_{i=1}^{N_m-1}\hat{d}_i} \tag{A3}$$

$$\tilde{a} = -h_m + \bar{c}h_m^3 + \sum_{i=1}^{N_m-1}(h_m - z_i)(\hat{d}_i\bar{c} - \hat{c}_i)$$

$$\tilde{b} = -\bar{c}h_m^3 - \sum_{i=1}^{N_m-1}(h_m - z_i)(\hat{d}_i\bar{c} - \hat{c}_i)$$

$$\tilde{c} = h_m^2 - \bar{b}h_m^3 - \sum_{i=1}^{N_m-1}(h_m - z_i)(\hat{d}_i\bar{b} - \hat{b}_i) \tag{A4}$$

$$\tilde{d} = \frac{1}{C_{55}^{(1)}}\left[h_m - \bar{a}h_m^3 - \sum_{i=1}^{N_m-1}(h_m - z_i)(\hat{d}_i\bar{a} - \hat{a}_i)\right]$$

$$\tilde{e} = \frac{\tilde{d}}{C_{55}^{(N_m)}}\left[h_m^3 + \sum_{i=1}^{N_m-1}(h_m - z_i)\hat{d}_i\right]$$


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